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Fermatean Fuzzy Aggregation Operators with Priority Degrees and their Applications

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ABSTRACT

Fermatean fuzzy numbers (FrFNs) have demonstrated significant utility in real-world scenarios for handling uncertain data. In this study, we focus on multi-criteria decision-making (MCDM) problems with prioritized parameters. To address this, we introduce the notion of "priority levels." By assigning non-negative real numbers, known as "priority degrees," to these priority levels, we establish aggregation operators (AOs). Our work puts forth a diverse set of prioritized operators, notably the Fermatean fuzzy prioritized averaging (FrFPA_d) operator with priority degrees and the Fermatean fuzzy prioritized geometric (FrFPG_d) operator with priority degrees. Through systematic comparisons, we highlight the superiority of our proposed methodology over other contemporary approaches already in use. We place particular emphasis on thoroughly investigating the influence of priority degrees on the overall decision-making outcomes. This analysis yields valuable insights into the implications and benefits of incorporating prioritization in MCDM. Furthermore, we provide a decision-making strategy based on the aforementioned operators, within the Fermatean fuzzy set environment. This strategy offers a practical framework for effective decision-making when faced with uncertainty.

Introduction

Decision-making is a fundamental cognitive process that permeates all facets of human existence, from personal decisions to intricate professional endeavors. It is my privilege as a professor to shed light on the significance and breadth of decision-making in various disciplines, emphasizing its crucial role in shaping outcomes, managing resources, and fostering progress.Success in the realms

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of business and management is predicated on the ability to make decisions. Effective leaders make a multitude of decisions every day, spanning from strategic planning to resource allocation and risk evaluation. In this context, decisions frequently involve weighing prospective profits against associated risks, optimizing operational efficiency, and anticipating market trends. In order to remain competitive and ensure sustainable development, it is crucial to be able to make sound decisions amidst uncertainty and a dynamic business environment. Engineering and technology disciplines are distinguished by complex problem-solving and innovation, requiring well-informed decisions at every stage of project development. Material selection, design decisions, and implementation strategies are examples of crucial decisions that engineers must make in order to achieve desired results, whether they are building a bridge, developing a new software system, or designing energy-efficient structures. These disciplines require a balance between performance, safety, cost-effectiveness, and environmental impact when making decisions. In healthcare and medicine, choices have substantial effects on patient health and treatment efficacy. To diagnose maladies, prescribe treatments, and determine appropriate interventions, medical professionals must rely on evidence-based decision making. Healthcare decisions are also heavily influenced by ethical considerations, as practitioners must evaluate potential benefits and risks while respecting patient autonomy and adhering to best practices and legal frameworks. Curriculum design, instructional methodologies, and student support mechanisms are influenced by decision-making in education and academia. Administrators and teachers must make decisions that promote a conducive learning environment, foster critical thinking, and meet the diverse requirements of students. In order to perpetually improve educational systems, evidence-based decisions in education involve the careful evaluation of learning outcomes and pedagogical approaches. In the realm of public policy and administration, decision-making influences the functioning of societies and the lives of millions. When drafting legislation and regulations, policymakers must address intricate social, economic, and environmental issues. In deliberations, empirical data are analyzed, stakeholders are consulted, and competing interests are balanced to produce policies that are effective, fair, and sustainable and serve the public interest. Decision-making is crucial to addressing ecological challenges and conserving natural resources within the context of environmental and sustainability sciences. Experts and policymakers must consider ecological impact, climate change, and sustainable resource management when evaluating the potential consequences of various actions. The objective of decisions in this field is to establish a balance between human needs and environmental preservation for the benefit of future generations.

Evidently, decision-making is a fundamental cognitive process that transcends all spheres of human activity. Its significance lies in guiding action, propelling development, and shaping the future. In every discipline, optimal solutions require decision-makers to navigate complexities, uncertainties, and ethical considerations. In the pursuance of knowledge, progress, and societal well-being, the scope of decision-making is unbounded, providing a common thread that binds diverse fields together. It was also revealed that it is impossible to model the operating conditions of human cognition mechanisms using simple data handling strategies that are based on crisp integers. As a direct consequence of using these methods, decision-makers (DMs) are left with murky conclusions and decisions that are not entirely transparent. As a consequence of this, DMs require a new ideology that enables them to interpret ambiguous data values and maintain their decision-making requirements according to the context. This is necessary in order for them to cope with confusing and fuzzy circumstances that occur in the world. In this sense, Zadeh has revolutionized the use of fuzzy set theory to represent ambiguous data [1]. Atanassov uncovered the concept of an intuitionistic fuzzy set (IFS) [2], and Yager produced a Pythagorean fuzzy set (PFS), which is an extension of IFS [3–5]. Senapati & Yager proposed the idea of Fermatean fuzzy sets (FrFSs) [6].

Data aggregation is vital for decision-making in many different domains, including corporate, administrative, social, medical, technological, psychological, and artificial intelligence. In the past, consciousness of the alternative was considered to be a discrete quantity or a linguistic number. On the other hand, because of the degree of uncertainty involved, the data cannot be easily aggregated. In point of fact, AOs play an important part in the context of MCDM difficulties, the primary objective of which is to arrive at a single number by combining a number of different inputs. Wang and Garg [7] proposed the idea of "Archimedean based Pythagorean fuzzy interactive" based operations and AOs with application to MCDM. Wang *et al.* [8] proposed the "Pythagorean fuzzy interactive Hamacher power" AOs with application to the assessment of express service quality. Huang *et al.* [9] initiated the idea of "Pythagorean fuzzy MULTIMOORA method based on distance measure and score function" with application to MCDM. Lin *et al.* [10] proposed the "directional correlation coefficient measures" for PFSs. Lin *et al.* [11] introduced the "correlation coefficient and entropy measures" for linguistic PFSs. Meng *et al.* [12] gave the idea of "knowledge diffusion trajectories in the Pythagorean fuzzy field based on main path analysis". Lin *et al.* [13] gave the "bibliometric analysis" for the PFSs. Chen *et al.* [14] proposed the framework of MCDM for the "sustainable building material selection". Chen *et al.* [15] also introduced the "expertise-based bid evaluation for construction-contractor selection with generalized comparative linguistic ELECTRE III". Chen *et al.* [16] proposed the newly idea of determining passenger demands and evaluating passenger satisfaction based on the online-review analysis. Wei and Lu [17] gave the idea of "Pythagorean fuzzy power AOs", Wu and Wei [18] presented the idea of "Pythagorean fuzzy Hamacher AOs" and Garg [19] proposed "confidence levels based Pythagorean fuzzy AOs" with application to MCDM. Qiyas *et al.* [20] introduced concept of "Yager operators with the picture fuzzy set environment and its application to emergency program selection". Linear Diophantine fuzzy soft-max AOs and numerically validated approach to modeling water hammer phenomena is given in [21, 22]. Several methods have been proposed for handling decision-making problems including q-rung orthopair fuzzy Aczel–Alsina AOs [23], q-rung orthopair fuzzy Einstein interactive geometric AOs [24].

The remaining portions of this article are organized as shown below. The concepts that are essential to FrFS are discussed in Section 2. In Section 3, we examined how the FrF prioritized AOs based on the priority vector and how effectively it was doing so. In Section 4, we present a method for resolving MCDM issues that is based on the introduction of new AOs. In Section 5, you can find a selection application for the agricultural land, as well as a comparative study with other methods. The conclusion of Section 6 includes some parting remarks as well as some suggestions for the future.

Certain fundamental concepts

In this particular section of the paper, we take into consideration some fundamental concepts and operational principles associated with FrFNs.

Definition 0.1. [6] Assume FrFS $\widetilde{\mathscr{T}}$ in $\mathcal{Q}$ is defined as

$$\widetilde{\mathscr{T}} = \{\langle \varsigma, \gimel^{\gamma}{}_{\widetilde{\mathscr{T}}}(\varsigma), \hbar^{\ell}{}_{\widetilde{\mathscr{T}}}(\varsigma)\rangle : \varsigma \in \mathcal{Q}\}$$

where $\gimel^{\gamma}{}_{\widetilde{\mathscr{T}}}, \hbar^{\ell}{}_{\widetilde{\mathscr{T}}} : \mathcal{Q} \to [0,1]$ defines the MSD and NMSD of the alternative $\varsigma \in \mathcal{Q}$ and $\forall \varsigma$ we have

$$0 \leq \gimel^{\gamma 3}_{\widetilde{\mathscr{T}}}(\varsigma) + \hbar^{\ell 3}_{\widetilde{\mathscr{T}}}(\varsigma) \leq 1.$$

Definition 0.2. [6] Let $\mathscr{T}^{\gimel}{}_1 = \langle \gimel^{\gamma}{}_1, \hbar^{\ell}{}_1\rangle$ and $\mathscr{T}^{\gimel}{}_2 = \langle \gimel^{\gamma}{}_2, \hbar^{\ell}{}_2\rangle$ be FrFNs. $\sigma > 0$, Then
(1) $\mathscr{T}^{\gimel c}_1 = \langle \hbar^{\ell}{}_1, \gimel^{\gamma}{}_1\rangle$
(2) $\mathscr{T}^{\gimel}{}_1 \vee \mathscr{T}^{\gimel}{}_2 = \langle max\{\gimel^{\gamma}{}_1, \hbar^{\ell}{}_1\}, min\{\gimel^{\gamma}{}_2, \hbar^{\ell}{}_2\}\rangle$
(3) $\mathscr{T}^{\gimel}{}_1 \wedge \mathscr{T}^{\gimel}{}_2 = \langle min\{\gimel^{\gamma}{}_1, \hbar^{\ell}{}_1\}, max\{\gimel^{\gamma}{}_2, \hbar^{\ell}{}_2\}\rangle$
(4) $\mathscr{T}^{\gimel}{}_1 \oplus \mathscr{T}^{\gimel}{}_2 = \langle \sqrt{(\gimel^{\gamma 3}_1 + \gimel^{\gamma 3}_2 - \gimel^{\gamma 3}_1 \gimel^{\gamma 3}_2)}, \hbar^{\ell}{}_1 \hbar^{\ell}{}_2\rangle$
(5) $\mathscr{T}^{\gimel}{}_1 \otimes \mathscr{T}^{\gimel}{}_2 = \langle \gimel^{\gamma}{}_1 \gimel^{\gamma}{}_2, \sqrt{(\hbar^{\ell 3}_1 + \hbar^{\ell 3}_2 - \hbar^{\ell 3}_1 \hbar^{\ell 3}_2)}\rangle$
(6) $\sigma \mathscr{T}^{\gimel}{}_1 = \langle \sqrt[3]{(1 - (1 - \gimel^{\gamma 3}_1)^{\sigma})}, \hbar^{\ell \sigma}_1\rangle$
(7)$\mathscr{T}^{\gimel \sigma}_1 = \langle \gimel^{\gamma \sigma}_1, \sqrt[3]{(1 - (1 - \hbar^{\ell 3}_1)^{\sigma})}\rangle$

Definition 0.3. [6] Let $\mathscr{T}^{\gimel} = \langle \gimel^{\gamma}, \hbar^{\ell}\rangle$ be the FrFN, score function Ξ of $\mathscr{T}^{\gimel}$ is defines as

$$\Xi(\mathscr{T}^{\gimel}) = \gimel^{\gamma 3} - \hbar^{\ell 3}$$

$\Xi(\mathscr{T}^{\gimel}) \in [-1, 1]$.

Definition 0.4. [6] Let $\mathscr{T}^{\gimel} = \langle \gimel^{\gamma}, \hbar^{\ell}\rangle$ be the FrFN, then an accuracy function H of $\mathscr{T}^{\gimel}$ is defines as

$$H(\mathscr{T}^{\gimel}) = \gimel^{\gamma 3} + \hbar^{\ell 3}$$

$H(\mathscr{T}^{\gimel}) \in [0, 1]$.

We introduce another score function, to support this type of research, $\mathscr{O}^{\varsigma}(R) = \frac{1 + \gimel^{\gamma 3}_{R} - \hbar^{\ell 3}_{R}}{2}$. We can see that $0 \leq \mathscr{O}^{\varsigma}(R) \leq 1$. This new score function satisfy all properties of score function defined by Senapati & Yager [6].

Definition 0.5. [6] Assume that $\mathscr{T}^{\daleth}{}_g = \langle \daleth^{\gamma}{}_g, \hbar^{\ell}{}_g \rangle$ is the conglomeration of FrFNs, and FrFWA: $\Lambda^n \to \Lambda$, if

$$\begin{aligned} FrFWA(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) &= \sum_{g=1}^{u} \hat{\mathfrak{H}}_g \mathscr{T}^{\daleth}{}_g \\ &= \hat{\mathfrak{H}}_1 \mathscr{T}^{\daleth}{}_1 \oplus \hat{\mathfrak{H}}_2 \mathscr{T}^{\daleth}{}_2 \oplus \ldots, \hat{\mathfrak{H}}_u \mathscr{T}^{\daleth}{}_u \end{aligned}$$

where Λ^n is the set of all FrFNs, and $\hat{\mathfrak{H}} = (\hat{\mathfrak{H}}_1, \hat{\mathfrak{H}}_2, \ldots, \hat{\mathfrak{H}}_u)^T$ is weight vector (WV), such that $0 \leqslant \hat{\mathfrak{H}}_u \leqslant 1$ and $\sum_{g=1}^{u} \hat{\mathfrak{H}}_u = 1$. Then, the FrFWA is called the "Fermatean weighted average operator".

Theorem 0.6. *[6] Let* $\mathscr{T}^{\daleth}{}_g = \langle \daleth^{\gamma}{}_g, \hbar^{\ell}{}_g \rangle$ be the conglomeration of FrFNs,we can find $FrFWG$ by

$$FrFWA(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \left\langle \sqrt[3]{(1 - \prod_{g=1}^{u} (1 - \daleth^{\gamma}{}^{3}_{g})^{\hat{\mathfrak{H}}_g})}, \prod_{g=1}^{u} \hbar^{\ell}{}_{g}^{\hat{\mathfrak{H}}_g} \right\rangle$$

Definition 0.7. [6] Assume that $\mathscr{T}^{\daleth}{}_g = \langle \daleth^{\gamma}{}_g, \hbar^{\ell}{}_g \rangle$ is the conglomeration of FrFN, and FrFWG:$\Lambda^n \to \Lambda$, if

$$\begin{aligned} FrFWG(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) &= \sum_{g=1}^{u} \mathscr{T}^{\daleth}{}_{g}^{\hat{\mathfrak{H}}_g} \\ &= \mathscr{T}^{\daleth}{}_{1}^{\hat{\mathfrak{H}}_1} \otimes \mathscr{T}^{\daleth}{}_{2}^{\hat{\mathfrak{H}}_2} \otimes \ldots, \mathscr{T}^{\daleth}{}_{u}^{\hat{\mathfrak{H}}_u} \end{aligned}$$

where Λ^n is the set of all FrFNs, and $\hat{\mathfrak{H}} = (\hat{\mathfrak{H}}_1, \hat{\mathfrak{H}}_2, \ldots, \hat{\mathfrak{H}}_u)^T$ is WV, such that $0 \leqslant \hat{\mathfrak{H}}_u \leqslant 1$ and $\sum_{g=1}^{u} \hat{\mathfrak{H}}_u = 1$. Then, the FrFWG is called the Fermatean weighted geometric operator.

Theorem 0.8. *[6] Let* $\mathscr{T}^{\daleth}{}_g = \langle \daleth^{\gamma}{}_g, \hbar^{\ell}{}_g \rangle$ be the conglomeration of FrFNs, we can find FrFWG by

$$FrFWG(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \left\langle \prod_{g=1}^{u} \daleth^{\gamma}{}_{g}^{\hat{\mathfrak{H}}_g}, \sqrt[3]{(1 - \prod_{g=1}^{u} (1 - \hbar^{\ell}{}^{3}_{g})^{\hat{\mathfrak{H}}_g})} \right\rangle \tag{0.1}$$

Definition 0.9. Let $\mathscr{T}^{\daleth}{}_g = \langle \daleth^{\gamma}{}_g, \hbar^{\ell}{}_g \rangle$ be the conglomeration of FrFNs, and FrFPWA:$\Lambda^n \to \Lambda$, be a n dimension mapping. if

$$\text{FrFPWA}(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \left(\frac{\breve{\beth}_1}{\sum_{g=1}^{u} \breve{\beth}_g} \mathscr{T}^{\daleth}{}_1 \oplus \frac{\breve{\beth}_2}{\sum_{g=1}^{u} \breve{\beth}_g} \mathscr{T}^{\daleth}{}_2 \oplus \ldots, \oplus \frac{\breve{\beth}_u}{\sum_{g=1}^{u} \breve{\beth}_g} \mathscr{T}^{\daleth}{}_u \right) \tag{0.2}$$

then the mapping FrFPWA is called Fermatean prioritized weighted averaging (FrFPWA) operator, where $\breve{\beth}_j = \prod_{k=1}^{j-1} \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_k)$ $(j = 2 \ldots, n)$, $\breve{\beth}_1 = 1$ and $\breve{\mathscr{O}}^{\varsigma}(\mathscr{T}^{\daleth}{}_k)$ is the score of k^{th} FrFN.

Definition 0.10. Let $\mathscr{T}^{\daleth}{}_p = \langle \daleth^{\gamma}{}_g, \hbar^{\ell}{}_p \rangle$ be the conglomeration of FrFNs, and FrFPWG:$\Lambda^n \to \Lambda$, be a n dimension mapping. if

$$\text{FrFPWG}(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \left(\mathscr{T}^{\daleth}{}_{1}^{\frac{\breve{\beth}_1}{\sum_{g=1}^{u} \breve{\beth}_g}} \otimes \mathscr{T}^{\daleth}{}_{2}^{\frac{\breve{\beth}_2}{\sum_{g=1}^{u} \breve{\beth}_g}} \otimes \ldots, \otimes \mathscr{T}^{\daleth}{}_{u}^{\frac{\breve{\beth}_u}{\sum_{g=1}^{u} \breve{\beth}_g}} \right) \tag{0.3}$$

then the mapping FrFPWG is called Fermatean prioritized weighted geometric (FrFPWG) operator, where $\breve{\beth}_j = \prod_{k=1}^{j-1} \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_k)$ $(j = 2 \ldots, n)$, $\breve{\beth}_1 = 1$ and $\breve{\mathscr{O}}^{\varsigma}(\mathscr{T}^{\daleth}{}_k)$ is the score of k^{th} FrFN.

Fermatean fuzzy prioritized aggregation operators with PDs

Within this section, we present the notion of FrFPA_d operator and FrFPG_d operator.

FrFPA_d operator

Let $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ be the conglomeration of FrFNs, there is a prioritization among these FrFNs expressed by the strict priority orders $\mathscr{T}^{\daleth}{}_1 \succ_{\text{II}_1} \mathscr{T}^{\daleth}{}_2 \succ_{\text{II}_2} \ldots \succ_{\text{II}_{u-1}} \mathscr{T}^{\daleth}{}_{u-1}$, where $\mathscr{T}^{\daleth}{}_u \succ_{\text{II}_u} \mathscr{T}^{\daleth}{}_{u+1}$ indicates that the FrFN $\mathscr{T}^{\daleth}{}_u$ has II_u higher priority than $\mathscr{T}^{\daleth}{}_{u+1}$. $\text{II} = (\text{II}_1, \text{II}_2, \ldots, \text{II}_{u-1})$ is the $(u-1)$ dimensional vector of PDs. The conglomeration of such FrFNs with strict priority orders and PDs is denoted by $\Re_d$.

Definition 0.11. A FrFPA_d operator is a mapping from $\Re_d^u$ to $\Re_d$ and defined as,

$$\text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_u) = \zeta_1^{(d)} \mathscr{T}^{\daleth}{}_1 \oplus \zeta_2^{(d)} \mathscr{T}^{\daleth}{}_2, \cdots, \zeta_u^{(d)} \mathscr{T}^{\daleth}{}_u \tag{0.4}$$

where $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}}$, $T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\text{II}_q}$, for each $g = (2, 3, \ldots, u)$ and $T_1 = 1$. Then FrFPA_d is called Fermatean prioritized averaging operators with PDs.

Theorem 0.12. *Assume $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs, we can also find $FrFPA_d$ by*

$$\begin{aligned} FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_u) &= \zeta_1^{(d)} \mathscr{T}^{\daleth}{}_1 \oplus \zeta_2^{(d)} \mathscr{T}^{\daleth}{}_2, \cdots, \zeta_u^{(d)} \mathscr{T}^{\daleth}{}_u \\ &= \sqrt[3]{1 - \prod_{g=1}^{u} (1 - \daleth^{\gamma 3}_{g})^{\zeta_g^{(d)}}}, \prod_{g=1}^{u} (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}} \end{aligned} \tag{0.5}$$

where $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}}$, $T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\text{II}_q}$, *for each* $g = (2, 3, \ldots, u)$ *and* $T_1 = 1$.

Proof. To prove this theorem, we use mathematical induction.
For $u = 2$

$$\zeta_1^{(d)} \mathscr{T}^{\daleth}{}_1 = \left(\sqrt[3]{1 - (1 - \daleth^{\gamma 3}_{1})^{\zeta_1^{(d)}}}, \hbar^{\ell}{}_1^{\zeta_1^{(d)}} \right)$$

$$\zeta_2^{(d)} \mathscr{T}^{\daleth}{}_2 = \left(\sqrt[3]{1 - (1 - \daleth^{\gamma 3}_{2})^{\zeta_2^{(d)}}}, \hbar^{\ell}{}_2^{\zeta_2^{(d)}} \right)$$

Then

$$\zeta_1^{(d)}\mathscr{T}^{\gimel}{}_1 \oplus \zeta_2^{(d)}\mathscr{T}^{\gimel}{}_2$$

$$= \left(\sqrt[3]{1-(1-\gimel\gamma_1^3)^{\zeta_1^{(d)}}}, \hbar^{\ell}{}_1^{\zeta_1^{(d)}}\right) \oplus \left(\sqrt[3]{1-(1-\gimel\gamma_2^3)^{\zeta_2^{(d)}}}, \hbar^{\ell}{}_2^{\zeta_2^{(d)}}\right)$$

$$= \left(\sqrt[3]{1-(1-\gimel\gamma_1^3)^{\zeta_1^{(d)}} + 1-(1-\gimel\gamma_2^3)^{\zeta_2^{(d)}} - \left(1-(1-\gimel\gamma_1^3)^{\zeta_1^{(d)}}\right)\left(1-(1-\gimel\gamma_2^3)^{\zeta_2^{(d)}}\right)}, \quad \hbar^{\ell}{}_1^{\zeta_1^{(d)}}.\hbar^{\ell}{}_2^{\zeta_2^{(d)}}\right)$$

$$= \left(\sqrt[3]{1-(1-\gimel\gamma_1^3)^{\zeta_1^{(d)}} + 1-(1-\gimel\gamma_2^3)^{\zeta_2^{(d)}} - \left(1-(1-\gimel\gamma_2^3)^{\zeta_2^{(d)}} - (1-\gimel\gamma_1^3)^{\zeta_1^{(d)}} + (1-\gimel\gamma_2^3)^{\zeta_1^{(d)}}(1-\gimel\gamma_1^3)^{\zeta_1^{(d)}}\right)}, \right.$$
$$\left. \hbar^{\ell}{}_1^{\zeta_1^{(d)}}.\hbar^{\ell}{}_2^{\zeta_2^{(d)}}\right)$$

$$= \left(\sqrt[3]{1-(1-\gimel\gamma_1^3)^{\zeta_1^{(d)}}(1-\gimel\gamma_2^3)^{\zeta_2^{(d)}}}, \hbar^{\ell}{}_1^{\zeta_1^{(d)}}.\hbar^{\ell}{}_2^{\zeta_2^{(d)}}\right)$$

$$= \left(\sqrt[3]{1-\prod_{g=1}^{3}(1-\gimel\gamma_g^3)^{\zeta_g^{(d)}}}, \prod_{g=1}^{u}(\hbar^{\ell}{}_g)^{\zeta_g^{(d)}}\right)$$

This shows that Equation 0.5 is true for $u=2$, now let that Equation 0.5 holds for $u=b$, i.e.,

$$\mathrm{FrFPA}_d(\mathscr{T}^{\gimel}{}_1, \mathscr{T}^{\gimel}{}_2, \dots \mathscr{T}^{\gimel}{}_b) = \left(\sqrt[3]{1-\prod_{g=1}^{b}(1-\gimel\gamma_g^3)^{\zeta_g^{(d)}}}, \prod_{g=1}^{b}\hbar^{\ell}{}_g^{\zeta_g^{(d)}}\right)$$

Now $u=b+1$, by operational laws of FrFNs we have,
$\mathrm{FrFPA}_d(\mathscr{T}^{\gimel}{}_1, \mathscr{T}^{\gimel}{}_2, \dots \mathscr{T}^{\gimel}{}_{b+1}) = \mathrm{FrFPA}_d(\mathscr{T}^{\gimel}{}_1, \mathscr{T}^{\gimel}{}_2, \dots \mathscr{T}^{\gimel}{}_b) \oplus \mathscr{T}^{\gimel}{}_{b+1}$

$$= \left(\sqrt[3]{1-\prod_{g=1}^{b}(1-\gimel\gamma_g^3)^{\zeta_g^{(d)}}}, \prod_{g=1}^{b}\hbar^{\ell}{}_g^{\zeta_g^{(d)}}\right) \oplus \left(\sqrt[3]{1-(1-\gimel\gamma_{b+1}^3)^{\zeta_{b+1}^{(d+1)}}}, \hbar^{\ell}{}_{b+1}^{\zeta_{b+1}^{(d+1)}}\right)$$

$$= \left(\sqrt[3]{1-\prod_{g=1}^{b}(1-\gimel\gamma_g^3)^{\zeta_g^{(d)}} + 1-(1-\gimel\gamma_{b+1}^3)^{\zeta_{b+1}^{(d+1)}} - \left(1-\prod_{g=1}^{b}(1-\gimel\gamma_g^3)^{\zeta_g^{(d)}}\right)\left(1-(1-\gimel\gamma_{b+1}^3)^{\zeta_{b+1}^{(d+1)}}\right)}, \prod_{g=1}^{b}\hbar^{\ell}{}_g^{\zeta_g^{(d)}}.\hbar^{\ell}{}_{b+1}^{\zeta_{b+1}^{(d+1)}}\right)$$

$$= \left(\sqrt[3]{1-\prod_{g=1}^{b+1}(1-\gimel\gamma_g^3)^{\zeta_g^{(d)}}}, \prod_{g=1}^{b+1}\hbar^{\ell}{}_g^{\zeta_g^{(d)}}\right)$$

This shows that for $u=b+1$, Equation 0.5 holds. Then,

$$\mathrm{FrFPA}_d(\mathscr{T}^{\gimel}{}_1, \mathscr{T}^{\gimel}{}_2, \dots \mathscr{T}^{\gimel}{}_u) = \left(\sqrt[3]{1-\prod_{g=1}^{u}(1-\gimel\gamma_g^3)^{\zeta_g^{(d)}}}, \prod_{g=1}^{u}\hbar^{\ell}{}_g^{\zeta_g^{(d)}}\right)$$

□

Furthermore, the suggested FrFPA_d operator is examined to ensure that it has idempotency and boundary properties. Their explanations are as follows:

Theorem 0.13. *Assume that* $\mathscr{T}^{\gimel}{}_g = (\gimel^{\gamma}{}_g, \hbar^{\ell}{}_g)$ *is the conglomeration of FrFNs, and*

$$\mathscr{T}^{\gimel-} = (min_g\,(\gimel^{\gamma}{}_g), max_g\,(\hbar^{\ell}{}_g)) \quad \text{and} \quad \mathscr{T}^{\gimel+} = (max_g\,(\gimel^{\gamma}{}_g), min_g\,(\hbar^{\ell}{}_g))$$

Then,

$$\mathscr{T}^{\daleth-} \leq FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_n) \leq \mathscr{T}^{\daleth+}$$

where $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}}$, $T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\amalg_q}$, *for each* $g = (2, 3, \ldots, u)$ *and* $T_1 = 1$.

Proof. Since,

$$min_g\ (\daleth^{\gamma}{}_g) \leq \daleth^{\gamma}{}_g \leq max_g\ (\daleth^{\gamma}{}_g) \tag{0.6}$$

and

$$min_g\ (\hbar^{\ell}{}_g) \leq \hbar^{\ell}{}_g \leq max_g\ (\hbar^{\ell}{}_g) \tag{0.7}$$

From Equation 0.6 we have,
$min_g\ (\daleth^{\gamma}{}_g) \leq \daleth^{\gamma}{}_g \leq max_g\ (\daleth^{\gamma}{}_g)$

$$\Leftrightarrow \sqrt[3]{min_g\ (\daleth^{\gamma}{}_g)^3} \leq \sqrt[3]{(\daleth^{\gamma}{}_g)^3} \leq \sqrt[3]{max_g\ (\daleth^{\gamma}{}_g)^3}$$

$$\Leftrightarrow \sqrt[3]{1 - max_g\ (\daleth^{\gamma}{}_g)^3} \leq \sqrt[3]{1 - (\daleth^{\gamma}{}_g)^3} \leq \sqrt[3]{1 - min_g\ (\daleth^{\gamma}{}_g)^3}$$

$$\Leftrightarrow \sqrt[3]{\left(1 - max_g\ (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq \sqrt[3]{\left(1 - (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq \sqrt[3]{\left(1 - min_g\ (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}}$$

$$\Leftrightarrow \sqrt[3]{\prod_{g=1}^{u}\left(1 - max_g\ (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq \sqrt[3]{\prod_{g=1}^{u}\left(1 - (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq \sqrt[3]{\prod_{g=1}^{u}\left(1 - min_g\ (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}}$$

$$\Leftrightarrow \sqrt[3]{1 - max_g\ (\daleth^{\gamma}{}_g)^3} \leq \sqrt[3]{\prod_{g=1}^{u}\left(1 - (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq \sqrt[3]{1 - min_g\ (\daleth^{\gamma}{}_g)^3}$$

$$\Leftrightarrow \sqrt[3]{-1 + min_j\ (\daleth^{\gamma}{}_g)^3} \leq \sqrt[3]{-\prod_{g=1}^{u}\left(1 - (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq \sqrt[3]{-1 + max_g\ (\daleth^{\gamma}{}_g)^3}$$

$$\Leftrightarrow \sqrt[3]{1 - 1 + min_j\ (\daleth^{\gamma}{}_g)^3} \leq \sqrt[3]{1 - \prod_{g=1}^{u}\left(1 - (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq \sqrt[3]{1 - 1 + max_g\ (\daleth^{\gamma}{}_g)^3}$$

$$\Leftrightarrow \sqrt[3]{min_j\ (\daleth^{\gamma}{}_g)^3} \leq \sqrt[3]{1 - \prod_{g=1}^{u}\left(1 - (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq \sqrt[3]{max_g\ (\daleth^{\gamma}{}_g)^3}$$

$$\Leftrightarrow min_j\ (\daleth^{\gamma}{}_g)^3 \leq \sqrt[3]{1 - \prod_{g=1}^{u}\left(1 - (\daleth^{\gamma}{}_g)^3\right)^{\zeta_g^{(d)}}} \leq max_g\ (\daleth^{\gamma}{}_g)^3$$

From Equation 0.7 we have,

$$min_g\ (\hbar^{\ell}{}_g) \leq \hbar^{\ell}{}_g \leq max_g\ (\hbar^{\ell}{}_g) \Leftrightarrow min_g\ (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}} \leq (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}} \leq max_g\ (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}}$$

$$\Leftrightarrow \prod_{g=1}^{u} min_g\ (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}} \leq \prod_{g=1}^{u} (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}} \leq \prod_{g=1}^{u} max_g\ (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}}$$

$$\Leftrightarrow min_g\ (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}} \leq \prod_{g=1}^{u} (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}} \leq max_g\ (\hbar^{\ell}{}_g)^{\zeta_g^{(d)}}$$

Let

$$\text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_n) = \mathscr{T}^{\daleth} = (\daleth^{\gamma}, \hbar^{\ell})$$

Then, $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}) = \daleth^{\gamma 3} - \hbar^{\ell^3} \quad \leq\ max_g\ (\daleth^{\gamma})^3 - min_j\ (\hbar^{\ell})^3 =\ \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{max})$ So, $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}) \leq\ \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{max})$.
Again, $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}) = \daleth^{\gamma 3} - \hbar^{\ell^3} \quad \geq\ min_g\ (\daleth^{\gamma})^3 - max_j\ (\hbar^{\ell})^3 =\ \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{min})$ So, $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}) \geq\ \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{min})$.
If, $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}) \leq\ \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{max})$ and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}) \geq\ \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{min})$, then

$$\mathscr{T}^{\daleth}{}_{min} \leq \text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_n) \leq \mathscr{T}^{\daleth}{}_{max} \tag{0.8}$$

If $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}) =\ \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{max})$, then $\daleth^{\gamma 3} - \hbar^{\ell^3} =\ max_g\ (\daleth^{\gamma})^3 - min_j\ (\hbar^{\ell})^3$

$$\Leftrightarrow \daleth^{\gamma 3} - \hbar^{\ell^3} =\ max_g\ (\daleth^{\gamma})^3 - min_g\ (\hbar^{\ell})^3$$
$$\Leftrightarrow \daleth^{\gamma 3} = max_g\ (\daleth^{\gamma})^3, \quad \hbar^{\ell^3} = min_g\ (\hbar^{\ell})^3$$
$$\Leftrightarrow \daleth^{\gamma} = max_g\ \daleth^{\gamma}, \quad \hbar^{\ell} = min_g\ \hbar^{\ell}$$

Now, $H(\mathscr{T}^{\daleth}) = \daleth^{\gamma 3} + \hbar^{\ell^3} = max_g\ (\daleth^{\gamma})^3 + min_g\ (\hbar^{\ell})^3 = H(\mathscr{T}^{\daleth}{}_{max})$

$$\text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_n) = \mathscr{T}^{\daleth}{}_{max} \tag{0.9}$$

If $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}) =\ \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{min})$, then $\daleth^{\gamma 3} - \hbar^{\ell^3} =\ min_g\ (\daleth^{\gamma})^3 - max_j\ (\hbar^{\ell})^3$

$$\Leftrightarrow \daleth^{\gamma 3} - \hbar^{\ell^3} =\ min_g\ (\daleth^{\gamma})^3 - max_g\ (\hbar^{\ell})^3$$
$$\Leftrightarrow \daleth^{\gamma 3} = min_g\ (\daleth^{\gamma})^3, \quad \hbar^{\ell^3} = max_g\ (\hbar^{\ell})^3$$
$$\Leftrightarrow \daleth^{\gamma} = min_g\ \daleth^{\gamma}, \quad \hbar^{\ell} = max_g\ \hbar^{\ell}$$

Now, $H(\mathscr{T}^{\daleth}) = \daleth^{\gamma 3} + \hbar^{\ell^3} = min_g\ (\daleth^{\gamma})^3 + max_g\ (\hbar^{\ell})^3 = H(\mathscr{T}^{\daleth}{}_{max})$

$$\text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_n) = \mathscr{T}^{\daleth}{}_{min} \tag{0.10}$$

Thus, from Equations 0.8, 0.9 and 0.10, we get

$$\mathscr{T}^{\daleth -} \leq \text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_n) \leq \mathscr{T}^{\daleth +}$$

□

Theorem 0.14. *Assume that if $\mathscr{T}^{\daleth}{}_{\diamond}$ is a FrFN satisfied the property, $\mathscr{T}^{\daleth}{}_g = \mathscr{T}^{\daleth}{}_{\diamond}$, $\forall g$ then*

$$FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \mathscr{T}^{\daleth}{}_{\diamond}$$

Proof. Let $\mathscr{T}^{\daleth}{}_{\diamond} = (\daleth^{\gamma}{\diamond}, \hbar^{\ell}{}_{\diamond})$ be the FrFN. Then by assumption, we have $\mathscr{T}^{\daleth}{}_{g} = \mathscr{T}^{\daleth}{}_{\diamond}$, $\forall g$ gives $\daleth^{\gamma}{}_{g} = \daleth^{\gamma}{\diamond}$ and $\hbar^{\ell}{}_{g} = \hbar^{\ell}{\diamond}$ $\forall g$. By Definition 0.11, we have $\sum_{g=1}^{u} \zeta_g^{(d)}$. Then by using Theorem 0.12, we get

$$
\begin{aligned}
\mathrm{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_u) &= \left(\sqrt[3]{1 - \prod_{g=1}^{u} (1 - \daleth^{\gamma}{}^{3}_{\diamond})^{\zeta_g^{(d)}}}, \prod_{g=1}^{u} \hbar^{\ell}{}_{\diamond}^{\zeta_g^{(d)}} \right) \\
&= \left(\sqrt[3]{1 - (1 - \daleth^{\gamma}{}^{3}_{\diamond})^{\sum_{g=1}^{u} \zeta_g^{(d)}}}, \hbar^{\ell}{}_{\diamond}^{\sum_{g=1}^{u} \zeta_g^{(d)}} \right) \\
&= (\daleth^{\gamma}{\diamond}, \hbar^{\ell}{}_{\diamond}) \\
&= \mathscr{T}^{\daleth}{}_{\diamond}
\end{aligned}
$$

□

Corollary 0.15. *If $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of largest FrFNs, i.e,. $\mathscr{T}^{\daleth}{}_g = (1, 0)$ for all g, then*

$$FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_u) = (1, 0)$$

Proof. We can easily obtain Corollary similar to the Theorem 0.14. □

Corollary 0.16. *If $\mathscr{T}^{\daleth}{}_1 = (\daleth^{\gamma}{}_1, \hbar^{\ell}{}_1)$ is the smallest FrFN, i.e,. $\mathscr{T}^{\daleth}{}_1 = (0, 1)$, then*

$$FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_u) = (0, 1)$$

Proof. Here, $\mathscr{T}^{\daleth}{}_1 = (0, 1)$ then by definition of the score function, we have,

$$\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1) = 0$$

Since,

$$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q) \right)^{\amalg_q}, \quad \text{for each} \quad g = (2, 3, \dots, u) \quad \text{and} \qquad T_1 = 1.$$

We have,

$$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q) \right)^{\amalg_q} = (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)^{\amalg_1}) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)^{\amalg_2}) \dots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1})^{\amalg_{g-1}}) = 0$$

From Definition 0.4, we have

$$
\begin{aligned}
\mathrm{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_u) &= \zeta_1^{(d)} \mathscr{T}^{\daleth}{}_1 \oplus \zeta_2^{(d)} \mathscr{T}^{\daleth}{}_2, \cdots, \zeta_u^{(d)} \mathscr{T}^{\daleth}{}_u \\
&= 1 \quad \mathscr{T}^{\daleth}{}_1 \oplus 0 \quad \mathscr{T}^{\daleth}{}_2 \oplus \dots 0 \quad \mathscr{T}^{\daleth}{}_n \\
&= \mathscr{T}^{\daleth}{}_1 = (0, 1)
\end{aligned}
$$

□

Theorem 0.17. *Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ and $\beta_g = (\phi_g, \varphi_g)$ are two conglomerations of FrFNs, if $r > 0$ and $\beta = (\daleth^{\gamma}{}_{\beta}, \hbar^{\ell}{}_{\beta})$ is a FrFN, then*

1. $FrFPA_d(\mathscr{T}^{\daleth}{}_1 \oplus \beta, \mathscr{T}^{\daleth}{}_2 \oplus \beta, \dots \mathscr{T}^{\daleth}{}_u \oplus \beta) = FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_u) \oplus \beta$

2. $FrFPA_d(r\mathscr{T}^{\daleth}{}_1, r\mathscr{T}^{\daleth}{}_2, \dots r\mathscr{T}^{\daleth}{}_u) = r \;\; FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_u)$

3.
$FrFPA_d(\mathscr{T}^{\daleth}{}_1 \oplus \beta_1, \mathscr{T}^{\daleth}{}_2 \oplus \beta_2, \dots \mathscr{T}^{\daleth}{}_u \oplus \beta_n) = FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_n) \oplus FrFPA_d(\beta_1, \beta_2, \dots \beta_u)$

4. $FrFPA_d(r\mathscr{T}^{\daleth}{}_1 \oplus \beta, r\mathscr{T}^{\daleth}{}_2 \oplus \beta, \dots \oplus r\mathscr{T}^{\daleth}{}_u \oplus \beta) = r \;\; FrFPA_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_u) \oplus \beta$

Proof. This is trivial by definition. □

FrFPA_d operator satisfied following properties.

Property:1

Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs, then we have

$$\lim_{(\mathrm{II}_1,\mathrm{II}_2,\ldots,\mathrm{II}_{u-1})\to(1,1,\ldots,1)} \text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \text{FrFPWA}(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) \tag{0.11}$$

Proof. Given that, $(\mathrm{II}_1, \mathrm{II}_2, \ldots, \mathrm{II}_{u-1}) \to (1, 1, \ldots, 1)$ from this we have,

$$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\mathrm{II}_q} \to \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right) = T_g$$

by this we obtain, $\zeta_g^{(d)} \to \zeta_g$

$\lim_{(\mathrm{II}_1,\mathrm{II}_2,\ldots,\mathrm{II}_{u-1})\to(1,1,\ldots,1)} \text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u)$

$$\begin{aligned} &= \lim_{(\mathrm{II}_1,\mathrm{II}_2,\ldots,\mathrm{II}_{u-1})\to(1,1,\ldots,1)} \zeta_1^{(d)} \mathscr{T}^{\daleth}{}_1 \oplus \zeta_2^{(d)} \mathscr{T}^{\daleth}{}_2, \cdots, \zeta_u^{(d)} \mathscr{T}^{\daleth}{}_u \\ &= \zeta_1 \mathscr{T}^{\daleth}{}_1 \oplus \zeta_2 \mathscr{T}^{\daleth}{}_2, \cdots, \zeta_u \mathscr{T}^{\daleth}{}_u \\ &= \text{FrFPWA}(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) \end{aligned}$$

□

Remark 0.18. When $\mathrm{II}_1 = \mathrm{II}_2 = \ldots, = \mathrm{II}_{u-1} = 1$, Property:1 states that the existing FrFPWA operator is a particular situation of the suggested FrFPA_d operator. As a result, FrFPA_d operator is more generic than FrFPWA operator.

Property:2

Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_g) \neq 0$ for all g, then we have

$$\lim_{(\mathrm{II}_1,\mathrm{II}_2,\ldots,\mathrm{II}_{u-1})\to(0,0,\ldots,0)} \text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \frac{1}{u}(\mathscr{T}^{\daleth}{}_1 \oplus \mathscr{T}^{\daleth}{}_2 \oplus, \ldots, \oplus \mathscr{T}^{\daleth}{}_u) \tag{0.12}$$

Proof. Given that, $(\mathrm{II}_1, \mathrm{II}_2, \ldots, \mathrm{II}_{u-1}) \to (0, 0, \ldots, 0)$ from this we have,

$$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\mathrm{II}_q} = 1$$

and $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}} = \frac{1}{n}$. Hence

$$\lim_{(\mathrm{II}_1,\mathrm{II}_2,\ldots,\mathrm{II}_{u-1})\to(0,0,\ldots,0)} \text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \frac{1}{u}(\mathscr{T}^{\daleth}{}_1 \oplus \mathscr{T}^{\daleth}{}_2 \oplus, \ldots, \oplus \mathscr{T}^{\daleth}{}_u)$$

□

Property:3

Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1) \neq 0$ or 1, then we have

$$\lim_{\mathrm{II}_1 \to +\infty} \text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \mathscr{T}^{\daleth}{}_1 \tag{0.13}$$

Proof. Here, $\mathrm{II}_1 \to +\infty$ for each $g = 2, 3, ..., u$ we have

$$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\mathrm{II}_q} = (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)^{+\infty}) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)^{\mathrm{II}_2}) \ldots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1})^{\mathrm{II}_{g-1}}) = 0$$

Because, $0 < \mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1) < 1$, $\sum_{g=1}^{u} T_g^{(d)} = T_1^{(d)} = 1 \Rightarrow \zeta_1^{(d)} = \frac{T_1^{(d)}}{\sum_{g=1}^{u} T_1^{(d)}} = 1$ and $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}}$

for each $g = 2, 3, ..., u$. Hence,

$$\lim_{\Pi_1 \to \infty} \mathrm{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \mathscr{T}^{\daleth}{}_1$$

□

Remark 0.19. According to Property:3, when $\Pi_1 \to +\infty$, the PD Π_1 of FrFN $\mathscr{T}^{\daleth}{}_1$ is very high in comparison to the PDs of other FrFNs. It indicates that FrFN $\mathscr{T}^{\daleth}{}_1$ is extremely essential. As a result, $\mathscr{T}^{\daleth}{}_1$ determines the aggregation result obtained by using the proposed operator FrFPA_d in this case.

Property:4
Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_g) \neq 0$ for all $g = 1, 2, ..., k+1$, and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{k+1}) \neq 1$ then we have

$$\lim_{(\Pi_1,\Pi_2,\ldots,\Pi_k,\Pi_{k+1}) \to (0,0,\ldots,0,+\infty)} \mathrm{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \frac{1}{k+1}(\mathscr{T}^{\daleth}{}_1 \oplus \mathscr{T}^{\daleth}{}_2 \oplus, \ldots, \oplus \mathscr{T}^{\daleth}{}_{k+1}) \tag{0.14}$$

Proof. Given that, $(\Pi_1, \Pi_2, \ldots, \Pi_k, \Pi_{k+1}) \to (0, 0, \ldots, 0, +\infty)$. So,
$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\Pi_q} = (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)^{\Pi_1}) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)^{\Pi_2}) \ldots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1})^{\Pi_{g-1}}) \to$
$(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1))^0 \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2))^0 \ldots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1}))^0 = 1$
for each $g = 2, 3, \ldots, k+1$.
$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\Pi_q} = (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)^{\Pi_1}) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)^{\Pi_2}) \ldots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1})^{\Pi_{g-1}}) \to$
$(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1))^0 \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2))^0 \ldots (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_k))^0 \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{k+1}))^{+\infty} \ldots (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1}))^{\Pi_{g-1}} = 0$
$\forall g = k+2, k+3, \ldots, u$
So,
$\sum_{g=1}^{u} T_g^{(d)} = T_1^{(d)} = k+1$ and $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}} \to \frac{1}{k+1}$ for each $g = 1, 2, 3, ..., k+1$.

$\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}} \to \frac{0}{k+1} = 0$ for each $g = k+2, k+3, \ldots, u$.
Hence,

$$\lim_{(\Pi_1,\Pi_2,\ldots,\Pi_k,\Pi_{k+1}) \to (0,0,\ldots,0,+\infty)} \mathrm{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \frac{1}{k+1}(\mathscr{T}^{\daleth}{}_1 \oplus \mathscr{T}^{\daleth}{}_2 \oplus, \ldots, \oplus \mathscr{T}^{\daleth}{}_{k+1})$$

□

Remark 0.20. When $(\Pi_1, \Pi_2, \ldots, \Pi_k, \Pi_{k+1}) \to (0, 0, \ldots, 0, +\infty)$, it means there's no prioritization association between the FrFNs $\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_{k+1}$ and that all of these FrFNs $\mathscr{T}^{\daleth}{}_1 \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_{k+1}$ have a much higher priority than the FrFNs $\mathscr{T}^{\daleth}{}_{k+2} \mathscr{T}^{\daleth}{}_{k+3}, \ldots, \mathscr{T}^{\daleth}{}_u$. As a result, the aggregated value is solely dependent on FrFNs $\mathscr{T}^{\daleth}{}_1 \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_{k+1}$, and these FrFNs $\mathscr{T}^{\daleth}{}_1 \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_{k+1}$ have similar weightage in the aggregation method.

Property:5
Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{k+1}) \neq 1$ or 0 then we have

$$\lim_{(\Pi_1,\Pi_2,\ldots,\Pi_k,\Pi_{k+1}) \to (1,1,\ldots,1,+\infty)} \mathrm{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \mathrm{FrFPWA}(\mathscr{T}^{\daleth}{}_1 \oplus \mathscr{T}^{\daleth}{}_2 \oplus, \ldots, \oplus \mathscr{T}^{\daleth}{}_{k+1}) \tag{0.15}$$

Proof. Given that, $(\Pi_1, \Pi_2, \ldots, \Pi_k, \Pi_{k+1}) \to (1, 1, \ldots, 1, +\infty)$. So,

$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\Pi_q} = (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)^{\Pi_1}) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)^{\Pi_2}) \ldots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1})^{\Pi_{g-1}}) \to (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)) \ldots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}$

for each $g = 2, 3, \ldots, k+1$.

$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\text{II}_q} = (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)^{\text{II}_1}) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)^{\text{II}_2}) \ldots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1})^{\text{II}_{g-1}}) \rightarrow$

$(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)) \ldots (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_k)) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{k+1}))^{+\infty} \ldots (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1}))^{\text{II}_{g-1}} = 0$

$\forall g = k+2, k+3, \ldots, u$

So,

$\sum_{g=1}^{u} T_g^{(d)} \rightarrow \sum_{g=1}^{k+1} T_g$ and $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}} \rightarrow \frac{T_g}{\sum_{g=1}^{k+1} T_g}$ for each $g = 1, 2, 3, ..., k+1$.

$\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}} \rightarrow \frac{0}{\sum_{g=1}^{k+1} T_g} = 0$ for each $g = k+2, k+3, \ldots, u$.

Hence,

$$\lim_{(\text{II}_1, \text{II}_2, \ldots, \text{II}_k, \text{II}_{k+1}) \rightarrow (1,1,\ldots,1,+\infty)} \text{FrFPA}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \text{FrFPWA}(\mathscr{T}^{\daleth}{}_1 \oplus \mathscr{T}^{\daleth}{}_2 \oplus, \ldots, \oplus \mathscr{T}^{\daleth}{}_{k+1})$$

□

Remark 0.21. When $(\text{II}_1, \text{II}_2, \ldots, \text{II}_k, \text{II}_{k+1}) \rightarrow (1, 1, \ldots, 1, +\infty)$, it means there's normal prioritization association between the FrFNs $\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_{k+1}$ and that all of these FrFNs $\mathscr{T}^{\daleth}{}_1 \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_{k+1}$ have a much higher priority than the FrFNs $\mathscr{T}^{\daleth}{}_{k+2} \mathscr{T}^{\daleth}{}_{k+3}, \ldots, \mathscr{T}^{\daleth}{}_u$. As a result, the aggregated value is solely dependent on FrFNs $\mathscr{T}^{\daleth}{}_1 \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_{k+1}$.

FrFPG$_d$ operator

Assume $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g) \quad (g = 1, 2..., u)$ is the conglomeration of FrFNs, there is a prioritization among these FrFNs expressed by the strict priority orders $\mathscr{T}^{\daleth}{}_1 \succ_{\text{II}_1} \mathscr{T}^{\daleth}{}_2 \succ_{\text{II}_2} \ldots \succ_{\text{II}_{u-1}} \mathscr{T}^{\daleth}{}_{u-1}$, where $\mathscr{T}^{\daleth}{}_u \succ_{\text{II}_u} \mathscr{T}^{\daleth}{}_{u+1}$ indicates that the FrFN $\mathscr{T}^{\daleth}{}_u$ has II_u higher priority than $\mathscr{T}^{\daleth}{}_{u+1}$. $d = (\text{II}_1, \text{II}_2, \ldots, \text{II}_{u-1})$ is the $(u-1)$ dimensional vector of PDs. The conglomeration of such FrFNs with strict priority orders and PDs is denoted by $\Re_d$.

Definition 0.22. A FrFPG$_d$ operator is a mapping from $\Re_d^u$ to $\Re_d$ and defined as,

$$\text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_u) = \mathscr{T}^{\daleth}{}_1^{\zeta_1^{(d)}} \otimes \mathscr{T}^{\daleth}{}_2^{\zeta_2^{(d)}}, \cdots, \mathscr{T}^{\daleth}{}_u^{\zeta_u^{(d)}} \tag{0.16}$$

where $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}}$, $T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\text{II}_q}$, for each $g = (2, 3, \ldots, u)$ and $T_1 = 1$. Then FrFPG$_d$ is called Fermatean prioritized geometric operator with PDs.

Theorem 0.23. *Assume $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs, we can also find FrFPG$_d$ by*

$$\begin{aligned} FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots, \mathscr{T}^{\daleth}{}_u) &= \mathscr{T}^{\daleth}{}_1^{\zeta_1^{(d)}} \otimes \mathscr{T}^{\daleth}{}_2^{\zeta_2^{(d)}}, \cdots, \mathscr{T}^{\daleth}{}_u^{\zeta_u^{(d)}} \\ &= \left(\prod_{g=1}^{u} (\daleth^{\gamma}{}_g)^{\zeta_g^{(d)}}, \sqrt[3]{1 - \prod_{g=1}^{u} (1 - \hbar^{\ell}{}_g^3)^{\zeta_g^{(d)}}} \right) \end{aligned} \tag{0.17}$$

where $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}}$, $T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\text{II}_q}$, *for each* $g = (2, 3, \ldots, u)$ *and* $T_1 = 1$.

Proof. To prove this theorem, we use mathematical induction.

For $u = 2$

$$\mathscr{T}^{\daleth}{}_1^{\zeta_1^{(d)}} = \left(\daleth^{\gamma}{}_1^{\zeta_1^{(d)}}, \sqrt[3]{1 - (1 - \hbar^{\ell}{}_1^3)^{\zeta_1^{(d)}}} \right)$$

$$\mathscr{T}^{\daleth}{}_2^{\zeta_2^{(d)}} = \left(\daleth^{\gamma}{}_2^{\zeta_2^{(d)}}, \sqrt[3]{1-(1-\hbar^{\ell}{}_2^3)^{\zeta_2^{(d)}}}\right)$$

Then

$\mathscr{T}^{\daleth}{}_1^{\zeta_1^{(d)}} \otimes \mathscr{T}^{\daleth}{}_2^{\zeta_2^{(d)}}$

$$= \left(\daleth^{\gamma}{}_1^{\zeta_1^{(d)}}, \sqrt[3]{1-(1-\hbar^{\ell}{}_1^3)^{\zeta_1^{(d)}}}\right) \otimes \left(\daleth^{\gamma}{}_2^{\zeta_2^{(d)}}, \sqrt[3]{1-(1-\hbar^{\ell}{}_2^3)^{\zeta_2^{(d)}}}\right)$$

$$= \left(\daleth^{\gamma}{}_1^{\zeta_1^{(d)}} . \daleth^{\gamma}{}_2^{\zeta_2^{(d)}}, \sqrt[3]{1-(1-\hbar^{\ell}{}_1^3)^{\zeta_1^{(d)}} + 1-(1-\hbar^{\ell}{}_2^3)^{\zeta_2^{(d)}} - \left(1-(1-\hbar^{\ell}{}_1^3)^{\zeta_1^{(d)}}\right)\left(1-(1-\hbar^{\ell}{}_2^3)^{\zeta_2^{(d)}}\right)}\right)$$

$$= \left(\daleth^{\gamma}{}_1^{\zeta_1^{(d)}} . \daleth^{\gamma}{}_2^{\zeta_2^{(d)}}, \sqrt[3]{1-(1-\hbar^{\ell}{}_1^3)^{\zeta_1^{(d)}} + 1-(1-\hbar^{\ell}{}_2^3)^{\zeta_2^{(d)}} - \left(1-(1-\hbar^{\ell}{}_2^3)^{\zeta_2^{(d)}} - (1-\hbar^{\ell}{}_1^3)^{\zeta_1^{(d)}} + (1-\hbar^{\ell}{}_2^3)^{\zeta_1^{(d)}} (1-\hbar^{\ell}{}_1^3)^{\zeta_1^{(d)}}\right)}\right)$$

$$= \left(\daleth^{\gamma}{}_1^{\zeta_1^{(d)}} . \daleth^{\gamma}{}_2^{\zeta_2^{(d)}}, \sqrt[3]{1-(1-\hbar^{\ell}{}_1^3)^{\zeta_1^{(d)}} (1-\hbar^{\ell}{}_2^3)^{\zeta_2^{(d)}}}\right)$$

$$= \left(\prod_{g=1}^{u} (\daleth^{\gamma}{}_g)^{\zeta_g^{(d)}}, \sqrt[3]{1-\prod_{g=1}^{3}(1-\hbar^{\ell}{}_g^3)^{\zeta_g^{(d)}}}\right)$$

This shows that Equation 0.17 is true for $u = 2$, now let that Equation 0.17 holds for $u = b$, i.e.,

$$\text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_b) = \left(\prod_{g=1}^{b} \daleth^{\gamma}{}_g^{\zeta_g^{(d)}}, \sqrt[3]{1-\prod_{g=1}^{b}(1-\hbar^{\ell}{}_g^3)^{\zeta_g^{(d)}}}\right)$$

Now $u = b + 1$, by operational laws of FrFNs we have,

$\text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_{b+1}) = \text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_b) \otimes \mathscr{T}^{\daleth}{}_{b+1}$

$$= \left(\prod_{g=1}^{b} \daleth^{\gamma}{}_g^{\zeta_g^{(d)}}, \sqrt[3]{1-\prod_{g=1}^{b}(1-\hbar^{\ell}{}_g^3)^{\zeta_g^{(d)}}}\right) \otimes \left(\daleth^{\gamma}{}_{b+1}^{\zeta_{b+1}^{(d+1)}}, \sqrt[3]{1-(1-\hbar^{\ell}{}_{b+1}^3)^{\zeta_{b+1}^{(d+1)}}}\right)$$

$$= \left(\prod_{g=1}^{b} \daleth^{\gamma}{}_g^{\zeta_g^{(d)}} . \daleth^{\gamma}{}_{b+1}^{\zeta_{b+1}^{(d+1)}}, \sqrt[3]{1-\prod_{g=1}^{b}(1-\hbar^{\ell}{}_g^3)^{\zeta_g^{(d)}} + 1-(1-\hbar^{\ell}{}_{b+1}^3)^{\zeta_{b+1}^{(d+1)}} - \left(1-\prod_{g=1}^{b}(1-\hbar^{\ell}{}_g^3)^{\zeta_g^{(d)}}\right)\left(1-(1-\hbar^{\ell}{}_{b+1}^3)^{\zeta_{b+1}^{(d+1)}}\right)}\right)$$

$$= \left(\prod_{g=1}^{b+1} \daleth^{\gamma}{}_g^{\zeta_g^{(d)}}, \sqrt[3]{1-\prod_{g=1}^{b+1}(1-\hbar^{\ell}{}_g^3)^{\zeta_g^{(d)}}}\right)$$

This shows that for $u = b + 1$, Equation 0.17 holds. Then,

$$\text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_u) = \left(\prod_{g=1}^{u} \daleth^{\gamma}{}_g^{\zeta_g^{(d)}}, \sqrt[3]{1-\prod_{g=1}^{u}(1-\hbar^{\ell}{}_g^3)^{\zeta_g^{(d)}}}\right)$$

□

Theorem 0.24. *Assume that* $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ *is the conglomeration of FrFNs, and*

$$\mathscr{T}^{\daleth -} = (min_g\ (\daleth^{\gamma}{}_g), max_g\ (\hbar^{\ell}{}_g)) \quad and \quad \mathscr{T}^{\daleth +} = (max_g\ (\daleth^{\gamma}{}_g), min_g\ (\hbar^{\ell}{}_g))$$

Then,

$$\mathscr{T}^{\daleth -} \leq FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \dots \mathscr{T}^{\daleth}{}_n) \leq \mathscr{T}^{\daleth +}$$

where $\zeta_g^{(d)} = \frac{T_g^{(d)}}{\sum_{g=1}^{u} T_g^{(d)}}$, $T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\text{II}_q}$, *for each* $g = (2, 3, \ldots, u)$ *and* $T_1 = 1$.

Proof. Proof is same as Theorem 0.13. □

Theorem 0.25. *Assume that if* $\mathscr{T}^{\daleth}{}_\diamond$ *is a FrFN satisfied the property,* $\mathscr{T}^{\daleth}{}_g = \mathscr{T}^{\daleth}{}_\diamond$, $\forall g$ *then*

$$FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \mathscr{T}^{\daleth}{}_\diamond$$

Proof. Let $\mathscr{T}^{\daleth}{}_\diamond = (\daleth^{\gamma}\diamond, \hbar^{\ell}{}_\diamond)$ be the FrFN. Then by assumption, we have $\mathscr{T}^{\daleth}{}_g = \mathscr{T}^{\daleth}{}_\diamond$, $\forall g$ gives $\daleth^{\gamma}{}_g = \daleth^{\gamma}\diamond$ and $\hbar^{\ell}{}_g = \hbar^{\ell}\diamond$ $\forall g$. By Definition 0.22, we have $\sum_{g=1}^{u} \zeta_g^{(d)}$. Then by using Theorem 0.23, we get

$$\begin{aligned}
\text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) &= \left(\prod_{g=1}^{u} \daleth^{\gamma}{}_\diamond^{\zeta_g^{(d)}}, \sqrt[3]{1 - \prod_{g=1}^{u}(1 - \hbar^{\ell}{}_\diamond^3)^{\zeta_g^{(d)}}}\right) \\
&= \left(\daleth^{\gamma}{}_\diamond^{\sum_{g=1}^{u} \zeta_g^{(d)}}, \sqrt[3]{1 - (1 - \hbar^{\ell}{}_\diamond^3)^{\sum_{g=1}^{u} \zeta_g^{(d)}}}\right) \\
&= (\daleth^{\gamma}\diamond, \hbar^{\ell}{}_\diamond) \\
&= \mathscr{T}^{\daleth}{}_\diamond
\end{aligned}$$

□

Corollary 0.26. *If* $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ *is the conglomeration of largest FrFNs, i.e,.* $\mathscr{T}^{\daleth}{}_g = (1, 0)$ *for all* g*, then*

$$FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = (1, 0)$$

Proof. We can easily obtain Corollary similar to the Theorem 0.25. □

Corollary 0.27. *If* $\mathscr{T}^{\daleth}{}_1 = (\daleth^{\gamma}{}_1, \hbar^{\ell}{}_1)$ *is the smallest FrFN, i.e,.* $\mathscr{T}^{\daleth}{}_1 = (0, 1)$*, then*

$$FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = (0, 1)$$

Proof. Here, $\mathscr{T}^{\daleth}{}_1 = (0, 1)$ then by definition of the score function, we have,

$$\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1) = 0$$

Since,

$$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\text{II}_q}, \quad \text{for each} \quad g = (2, 3, \ldots, u) \quad \text{and} \qquad T_1 = 1.$$

We have,

$$T_g^{(d)} = \prod_{q=1}^{g-1} \left(\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_q)\right)^{\text{II}_q} = (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1)^{\text{II}_1}) \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_2)^{\text{II}_2}) \ldots \quad (\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{g-1})^{\text{II}_{g-1}}) = 0$$

From Definition 0.4, we have

$$\begin{aligned}
\text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) &= \mathscr{T}^{\daleth}{}_1^{\zeta_1^{(d)}} \otimes \mathscr{T}^{\daleth}{}_2^{\zeta_2^{(d)}}, \cdots, \mathscr{T}^{\daleth}{}_u^{\zeta_u^{(d)}} \\
&= \mathscr{T}^{\daleth}{}_1^{1} \otimes \quad \mathscr{T}^{\daleth}{}_2^{0} \otimes \ldots \quad \mathscr{T}^{\daleth}{}_u^{0} \\
&= \mathscr{T}^{\daleth}{}_1 = (0, 1)
\end{aligned}$$

□

Theorem 0.28. *Assume that* $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ *and* $\beta_g = (\phi_g, \varphi_g)$ *are two conglomerations of FrFNs, if* $r > 0$ *and* $\beta = (\daleth^{\gamma}{}_{\beta}, \hbar^{\ell}{}_{\beta})$ *is a FrFN, then*
1. $FrFPG_d(\mathscr{T}^{\daleth}{}_1 \oplus \beta, \mathscr{T}^{\daleth}{}_2 \oplus \beta, \ldots \mathscr{T}^{\daleth}{}_u \oplus \beta) = FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) \oplus \beta$
2. $FrFPG_d(r\mathscr{T}^{\daleth}{}_1, r\mathscr{T}^{\daleth}{}_2, \ldots r\mathscr{T}^{\daleth}{}_u) = r\ \ FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u)$
3. $FrFPG_d(\mathscr{T}^{\daleth}{}_1 \oplus \beta_1, \mathscr{T}^{\daleth}{}_2 \oplus \beta_2, \ldots \mathscr{T}^{\daleth}{}_u \oplus \beta_n) = FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_n) \oplus FrFPG_d(\beta_1, \beta_2, \ldots \beta_u)$
4. $FrFPG_d(r\mathscr{T}^{\daleth}{}_1 \oplus \beta, r\mathscr{T}^{\daleth}{}_2 \oplus \beta, \ldots \oplus r\mathscr{T}^{\daleth}{}_u \oplus \beta) = r\ \ FrFPG_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) \oplus \beta$

Proof. This is trivial by definition. □

FrFPG_d operator also satisfied following properties.
Property:1
Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs, then we have

$$\lim_{(\text{Ш}_1,\text{Ш}_2,\ldots,\text{Ш}_{u-1})\to(1,1,\ldots,1)} \text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \text{FrFPWG}(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) \tag{0.18}$$

Property:2
Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_g) \neq 0$ for all g, then we have

$$\lim_{(\text{Ш}_1,\text{Ш}_2,\ldots,\text{Ш}_{u-1})\to(0,0,\ldots,0)} \text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \frac{1}{u}(\mathscr{T}^{\daleth}{}_1 \otimes \mathscr{T}^{\daleth}{}_2 \otimes, \ldots, \otimes \mathscr{T}^{\daleth}{}_u) \tag{0.19}$$

Property:3
Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_1) \neq 0$ or 1, then we have

$$\lim_{\text{Ш}_1\to+\infty} \text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \mathscr{T}^{\daleth}{}_1 \tag{0.20}$$

Property:4
Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_g) \neq 0$ for all $g = 1, 2, ..., k+1$, and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{k+1}) \neq 1$ then we have

$$\lim_{(\text{Ш}_1,\text{Ш}_2,\ldots,\text{Ш}_k,\text{Ш}_{k+1})\to(0,0,\ldots,0,+\infty)} \text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \frac{1}{k+1}(\mathscr{T}^{\daleth}{}_1 \otimes \mathscr{T}^{\daleth}{}_2 \otimes, \ldots, \otimes \mathscr{T}^{\daleth}{}_{k+1}) \tag{0.21}$$

Property:5
Assume that $\mathscr{T}^{\daleth}{}_g = (\daleth^{\gamma}{}_g, \hbar^{\ell}{}_g)$ is the conglomeration of FrFNs and $\mathscr{O}^{\varsigma}(\mathscr{T}^{\daleth}{}_{k+1}) \neq 1$ or 0 then we have

$$\lim_{(\text{Ш}_1,\text{Ш}_2,\ldots,\text{Ш}_k,\text{Ш}_{k+1})\to(1,1,\ldots,1,+\infty)} \text{FrFPG}_d(\mathscr{T}^{\daleth}{}_1, \mathscr{T}^{\daleth}{}_2, \ldots \mathscr{T}^{\daleth}{}_u) = \text{FrFPWG}(\mathscr{T}^{\daleth}{}_1 \otimes \mathscr{T}^{\daleth}{}_2 \otimes, \ldots, \otimes \mathscr{T}^{\daleth}{}_{k+1}) \tag{0.22}$$

Methodology for MCDM using proposed AOs

Let $\mathscr{A}^{\xi} = \{\mathscr{A}^{\xi}{}_1, \mathscr{A}^{\xi}{}_2, \ldots, \mathscr{A}^{\xi}{}_m\}$ be the conglomeration of alternatives and $\mathscr{C}^{\varsigma} = \{\mathscr{C}^{\varsigma}{}_1, \mathscr{C}^{\varsigma}{}_2, \ldots, \mathscr{C}^{\varsigma}{}_n\}$ is the conglomeration of criterions, priorities are assigned between the criterions provided by strict priority orientation. $\mathscr{C}^{\varsigma}{}_1 \succ_{\text{Ш}_1} \mathscr{C}^{\varsigma}{}_2 \succ_{\text{Ш}_2} \mathscr{C}^{\varsigma}{}_3 \ldots \succ_{\text{Ш}_{n-1}} \mathscr{C}^{\varsigma}{}_n$, indicates criteria $\mathscr{C}^{\varsigma}{}_J$ has a high priority than $\mathscr{C}^{\varsigma}{}_{J+1}$ with degree Ш_q for $q \in \{1, 2, \ldots, (n-1)\}$.
$K = \{K_1, K_2, \ldots, K_p\}$ is a conglomeration of decision-makers (DMs). Priorities are assigned between the DMs provided by strict priority orientation, $K_1 \succ_{\text{Ш}'_1} K_2 \succ_{\text{Ш}'_2} K_3 \ldots \succ_{\text{Ш}'_{p-1}} K_p$. DMs give a matrix according to their own standpoints $D^{(p)} = (\mathscr{B}_{ij}^{(p)})_{m\times n}$, where $\mathscr{B}_{ij}^{(p)}$ is given for the alternatives $\mathscr{A}^{\xi}{}_i \in \mathscr{A}^{\xi}$ with respect to the attribute $\mathscr{C}^{\varsigma}{}_j \in \mathscr{C}^{\varsigma}$ by K_p DM. If all Performance criteria are the same kind, there is no need for normalization; however, since MCGDM has two different types of Evaluation criteria (benefit kind attributes τ_b and cost kinds attributes τ_c), the matrix $D_(p)$ has been transformed into a normalize matrix using the normalization formula $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$,

$$(\mathscr{P}_{ij}^{(p)})_{m\times n} = \begin{cases} (\mathscr{B}_{ij}^{(p)})^c; & j \in \tau_c \\ \mathscr{B}_{ij}^{(p)}; & j \in \tau_b. \end{cases} \tag{0.23}$$

where $(\mathscr{B}_{ij}^{(p)})^c$ show the compliment of $\mathscr{B}_{ij}^{(p)}$.
The suggested operators will be implemented to the MCGDM, which will require the preceding steps.

Algorithm

Step 1:
Obtain the decision matrix $D^{(p)} = (\mathscr{B}_{ij}^{(p)})_{m\times n}$ in the format of FrFNs from DMs.

$$\begin{array}{cc} & \begin{array}{cccc} \mathscr{C}^{\varsigma}{}_1 & \mathscr{C}^{\varsigma}{}_2 & & \mathscr{C}^{\varsigma}{}_n \end{array} \\ \begin{array}{cc} K_1 & \mathscr{A}^{\xi}{}_1 \\ & \mathscr{A}^{\xi}{}_2 \\ & \vdots \\ & \mathscr{A}^{\xi}{}_m \\ K_2 & \mathscr{A}^{\xi}{}_1 \\ \vdots & \mathscr{A}^{\xi}{}_2 \\ & \vdots \\ & \mathscr{A}^{\xi}{}_m \\ K_p & \mathscr{A}^{\xi}{}_1 \\ & \mathscr{A}^{\xi}{}_2 \\ & \vdots \\ & \mathscr{A}^{\xi}{}_m \end{array} & \begin{bmatrix} (\daleth\gamma^1_{11}, \hbar\ell^1_{11}) & (\daleth\gamma^1_{12}, \hbar\ell^1_{12}) & \cdots\cdots & (\daleth\gamma^1_{1n}, \hbar\ell^1_{1n}) \\ (\daleth\gamma^1_{21}, \hbar\ell^1_{21}) & (\daleth\gamma^1_{22}, \hbar\ell^1_{22}) & \cdots\cdots & (\daleth\gamma^1_{2n}, \hbar\ell^1_{2n}) \\ \vdots & \vdots & \ddots\ddots & \vdots \\ (\daleth\gamma^1_{m1}, \hbar\ell^1_{m1}) & (\daleth\gamma^1_{m2}, \hbar\ell^1_{m2}) & \cdots\cdots & (\daleth\gamma^1_{mn}, \hbar\ell^1_{mn}) \\ (\daleth\gamma^3_{11}, \hbar\ell^3_{11}) & (\daleth\gamma^3_{12}, \hbar\ell^3_{12}) & \cdots\cdots & (\daleth\gamma^3_{1n}, \hbar\ell^3_{1n}) \\ (\daleth\gamma^3_{21}, \hbar\ell^3_{21}) & (\daleth\gamma^3_{22}, \hbar\ell^3_{22}) & \cdots\cdots & (\daleth\gamma^3_{2n}, \hbar\ell^3_{2n}) \\ \vdots & \vdots & \ddots\ddots & \vdots \\ (\daleth\gamma^3_{m1}, \hbar\ell^3_{m1}) & (\daleth\gamma^3_{m2}, \hbar\ell^3_{m2}) & \cdots\cdots & (\daleth\gamma^3_{mn}, \hbar\ell^3_{mn}) \\ (\daleth\gamma^p_{11}, \hbar\ell^p_{11}) & (\daleth\gamma^p_{12}, \hbar\ell^p_{12}) & \cdots\cdots & (\daleth\gamma^p_{1n}, \hbar\ell^p_{1n}) \\ (\daleth\gamma^p_{21}, \hbar\ell^p_{21}) & (\daleth\gamma^p_{22}, \hbar\ell^p_{22}) & \cdots\cdots & (\daleth\gamma^p_{2n}, \hbar\ell_{2n})^p \\ \vdots & \vdots & \ddots\ddots & \vdots \\ (\daleth\gamma^p_{m1}, \hbar\ell^p_{m1}) & (\daleth\gamma^p_{m2}, \hbar\ell^p_{m2}) & \cdots\cdots & (\daleth\gamma^p_{mn}, \hbar\ell^p_{mn}) \end{bmatrix} \end{array}$$

Step 2:
Two kinds of criterion are described in the decision matrix: (τ_c) cost type indicators and (τ_b) benefit type indicators. There is no need for normalisation if all indicators are of the same kind, but in MCGDM, there may be two types of criteria. The matrix was updated to the transforming response matrix in this case $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$ using the normalization formula Equation 0.23

Step 3:
Using one of provided AOs to combine all of the independent FrF-decision matrices $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$ into one combined evaluation matrix of the alternatives $W^{(p)} = (\widetilde{\chi}_{ij})_{m\times n}$.
$\widetilde{\chi}_{ij} = \text{FrFPA}_d(\mathscr{P}_{ij}^{(1)}, \mathscr{P}_{ij}^{(2)}, \dots \mathscr{P}_{ij}^{(p)})$

$$= \left(\sqrt[3]{1 - \prod_{z=1}^{p} \left(1 - ((\daleth\gamma^z_{ij})^3\right)^{\zeta_{ij}^{(z)}}}, \prod_{z=1}^{p} (\hbar\ell^z_{ij})^{\zeta_{ij}^{(z)}} \right) \tag{0.24}$$

or
$\widetilde{\chi}_{ij} = \text{FrFPG}_d(\mathscr{P}_{ij}^{(1)}, \mathscr{P}_{ij}^{(2)}, \dots \mathscr{P}_{ij}^{(p)})$

$$= \left(\prod_{z=1}^{p} (\daleth\gamma^z_{ij})^{\zeta_{ij}^{(z)}}, \sqrt[3]{1 - \prod_{z=1}^{p} \left(1 - ((\hbar\ell^z_{ij})^3\right)^{\zeta_{ij}^{(z)}}} \right) \tag{0.25}$$

Step 4:
Aggregate the FrF-values $\widetilde{\chi}_{ij}$ for each alternative $\mathscr{A}^{\xi}{}_i$ by the FrFPA_d (or FrFPG_d) operator.
$\widetilde{\chi}_{ij} = \text{FrFPA}_d(\mathscr{P}_{i1}, \mathscr{P}_{i2}, \dots \mathscr{P}_{in})$

$$= \left(\sqrt[3]{1 - \prod_{j=1}^{n} \left(1 - \daleth\gamma^3_{ij}\right)^{\zeta_{ij}}}, \prod_{j=1}^{n} (\hbar\ell_{ij})^{\zeta_{ij}} \right) \tag{0.26}$$

or
$\widetilde{\chi}_{ij} = \text{FrFPG}_d(\mathscr{P}_{i1}, \mathscr{P}_{i2}, \dots \mathscr{P}_{in})$

$$= \left(\prod_{j=1}^{n} (\daleth\gamma_{ij})^{\zeta_{ij}}, \sqrt[3]{1 - \prod_{j=1}^{n} \left(1 - \hbar\ell^3_{ij}\right)^{\zeta_{ij}}} \right) \tag{0.27}$$

Step 5:
Analyze the score for all cumulative alternative assessments.
Step 6:
The alternatives were classified by the score function and the most suitable alternative was selected.

Case study

This case study aims to demonstrate the significance of material selection in engineering design, particularly when confronted with multiple criteria, including cost. An engineering team is entrusted with selecting the most suitable material for a crucial component of a heavy-duty industrial machine in a hypothetical scenario. The evaluation of the alternatives is based on four criteria: mechanical properties, environmental impact, manufacturability, and cost. The objective of the team's analytical approach is to arrive at a decision that optimizes the component's performance while keeping costs in control. The engineering team is developing a crucial component for a mining industry heavy-duty industrial machine. The material selection process for this component plays a crucial role in determining the final product's overall performance, durability, and cost-effectiveness. In order to attain an optimal performance-to-cost ratio, the team has identified four key criteria: mechanical properties, environmental impact, manufacturability, and cost. Cost is a particularly important criterion, as the undertaking has a strict budget constraint.

- Mechanical Properties: The mechanical properties of the material are fundamental in ensuring the component's ability to withstand heavy loads, high temperatures, and harsh operating conditions. The team will evaluate tensile strength, hardness, fatigue resistance, and other relevant properties to ensure the chosen material can meet or exceed the required performance standards.
- Cost: Cost is a critical factor in this decision-making process. The team needs to strike a balance between material performance and affordability. The material's cost per unit will be compared to the overall budget and weighed against the other criteria to determine the most cost-effective option.
- Manufacturability: The ease of manufacturing and processing the material into the required shape and specifications is crucial to maintaining efficient production processes. The team will analyze factors such as machinability, weldability, and forming capabilities of each alternative.
- Environmental Impact: Considering the growing concern for environmental sustainability, it is imperative to assess the environmental impact of each material option. Factors such as recyclability, carbon footprint, and the presence of hazardous substances will be evaluated to minimize the environmental footprint of the chosen material.
- Surface Finish Quality: The surface finish quality criterion assesses the final appearance and texture of the component's surface. A smooth and aesthetically appealing surface is desirable for certain applications, especially when exposed to customers or end-users.

Alternatives:

- Steel Alloy (${\mathscr{A}^{\xi}}_1$): Steel Alloy A is a commonly used material known for its excellent mechanical properties and cost-effectiveness. It has high tensile strength, good fatigue resistance, and is relatively easy to manufacture. However, its environmental impact and recyclability need to be thoroughly assessed.
- Aluminum Alloy (${\mathscr{A}^{\xi}}_2$): Aluminum Alloy B boasts high strength-to-weight ratio, good corrosion resistance, and recyclability. Nevertheless, it may fall short in mechanical properties when compared to steel. The higher cost of aluminum is a potential drawback.
- Composite Material (${\mathscr{A}^{\xi}}_3$): Composite Material C offers the potential for exceptional mechanical properties and weight reduction. It may also exhibit a lower environmental impact due to reduced material usage. However, its manufacturability and cost may pose challenges.
- Cast Iron (${\mathscr{A}^{\xi}}_4$): Cast Iron D is a classic material known for its excellent mechanical properties, low cost, and ease of casting. However, it may have a higher environmental impact, and its weight could be a concern for certain applications.

Numerical illustration

Consider a decision making problem of finding out the most appropriate material selection. Assume the conglomeration of alternatives, $\mathscr{A}^{\xi}{}_1$, $\mathscr{A}^{\xi}{}_2$, $\mathscr{A}^{\xi}{}_3$ and $\mathscr{A}^{\xi}{}_4$ given as above. There are five criterions for evaluation of these alternatives $\mathscr{C}^{\varsigma}{}_1$= mechanical properties, $\mathscr{C}^{\varsigma}{}_2$= cost, $\mathscr{C}^{\varsigma}{}_3$= manufacturability, $\mathscr{C}^{\varsigma}{}_4$= environmental impact and $\mathscr{C}^{\varsigma}{}_5$= surface finish quality. Assume that the criterions have been prioritized in strict priority order $\mathscr{C}^{\varsigma}{}_1 >_{\amalg_1} \mathscr{C}^{\varsigma}{}_2 >_{\amalg_2} \mathscr{C}^{\varsigma}{}_3 >_{\amalg_3} \mathscr{C}^{\varsigma}{}_4 >_{\amalg_1} \mathscr{C}^{\varsigma}{}_5$. The three dimensional vector of PD is d = (2, 1, 3, 2). Here three DMs K_1, K_2 and K_3 are involved, they have been prioritized in strict priority order $K_1 >_{\amalg'_1} K_2 >_{\amalg'_2} K_3$, where d' = (3, 4).

~~Algorithm~~

Step 1:

Obtain the decision matrix $D^{(p)} = (\mathscr{B}_{ij}^{(p)})_{m\times n}$ in the format of FrFNs from DMs. The judgement values, given by three DMs, are described in Table 2.

Table 2: Rating given by DMs

Experts	Alternatives	$\mathscr{C}^{\varsigma}{}_1$	$\mathscr{C}^{\varsigma}{}_2$	$\mathscr{C}^{\varsigma}{}_3$	$\mathscr{C}^{\varsigma}{}_4$	$\mathscr{C}^{\varsigma}{}_5$
K_1	$\mathscr{A}^{\xi}{}_1$	(0.644, 0.246)	(0.244, 0.721)	(0.737, 0.172)	(0.832, 0.244)	(0.831, 0.129)
	$\mathscr{A}^{\xi}{}_2$	(0.595, 0.180)	(0.422, 0.711)	(0.433, 0.647)	(0.612, 0.320)	(0.520, 0.322)
	$\mathscr{A}^{\xi}{}_3$	(0.445, 0.420)	(0.145, 0.462)	(0.622, 0.421)	(0.582, 0.426)	(0.515, 0.138)
	$\mathscr{A}^{\xi}{}_4$	(0.235, 0.290)	(0.344, 0.693)	(0.437, 0.240)	(0.455, 0.347)	(0.527, 0.239)
K_2	$\mathscr{A}^{\xi}{}_1$	(0.764, 0.415)	(0.487, 0.235)	(0.853, 0.235)	(0.873, 0.168)	(0.844, 0.443)
	$\mathscr{A}^{\xi}{}_2$	(0.563, 0.129)	(0.534, 0.839)	(0.387, 0.728)	(0.131, 0.762)	(0.167, 0.259)
	$\mathscr{A}^{\xi}{}_3$	(0.483, 0.247)	(0.566, 0.623)	(0.290, 0.140)	(0.827, 0.657)	(0.573, 0.837)
	$\mathscr{A}^{\xi}{}_4$	(0.237, 0.964)	(0.460, 0.926)	(0.779, 0.868)	(0.555, 0.735)	(0.783, 0.925)
K_3	$\mathscr{A}^{\xi}{}_1$	(0.288, 0.822)	(0.677, 0.225)	(0.373, 0.183)	(0.877, 0.221)	(0.384, 0.213)
	$\mathscr{A}^{\xi}{}_2$	(0.276, 0.174)	(0.422, 0.753)	(0.427, 0.643)	(0.644, 0.336)	(0.517, 0.346)
	$\mathscr{A}^{\xi}{}_3$	(0.724, 0.456)	(0.425, 0.469)	(0.619, 0.421)	(0.528, 0.422)	(0.537, 0.353)
	$\mathscr{A}^{\xi}{}_4$	(0.426, 0.231)	(0.421, 0.671)	(0.473, 0.228)	(0.427, 0.323)	(0.674, 0.266)

Step 2:

Normalize the decision matrixes acquired by DMs using Equation 0.23. In Table 2, there are two types of criterions. C_2 is cost type criteria and others are benefit type criterions. Normalized FrF decision matrix given in Table 3.

Step 3:

Using FrFPA$_d$ opeartor to combine all of the independent FrFdecision matrices $Y^{(p)} = (\mathscr{P}_{ij}^{(p)})_{m\times n}$ into one combined evaluation matrix of the alternatives $W^{(p)} = (\widetilde{\chi}_{ij})_{m\times n}$ given in Table 4. First we find $T_{ij}^{(1)}$ $T_{ij}^{(2)}$ and $T_{ij}^{(3)}$, which are used in the calculation of FrFPA$_d$ operator.

$$T_{ij}^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Table 3: Normalized FrF decision matrix

Experts	**Alternatives**	$\mathscr{C}^{\varsigma}{}_{\mathbf{1}}$	$\mathscr{C}^{\varsigma}{}_{\mathbf{2}}$	$\mathscr{C}^{\varsigma}{}_{\mathbf{3}}$	$\mathscr{C}^{\varsigma}{}_{\mathbf{4}}$	$\mathscr{C}^{\varsigma}{}_{\mathbf{5}}$
K_1	$\mathscr{A}^{\xi}{}_1$	(0.644, 0.246)	(0.721, 0.244)	(0.737, 0.172)	(0.832, 0.244)	(0.831, 0.129)
	$\mathscr{A}^{\xi}{}_2$	(0.595, 0.180)	(0.711, 0.422)	(0.433, 0.647)	(0.612, 0.320)	(0.520, 0.322)
	$\mathscr{A}^{\xi}{}_3$	(0.445, 0.420)	(0.462, 0.145)	(0.622, 0.421)	(0.582, 0.426)	(0.515, 0.138)
	$\mathscr{A}^{\xi}{}_4$	(0.235, 0.290)	(0.693, 0.344)	(0.437, 0.240)	(0.455, 0.347)	(0.527, 0.239)
K_2	$\mathscr{A}^{\xi}{}_1$	(0.764, 0.415)	(0.235, 0.487)	(0.853, 0.235)	(0.873, 0.168)	(0.844, 0.443)
	$\mathscr{A}^{\xi}{}_2$	(0.563, 0.129)	(0.839, 0.534)	(0.387, 0.728)	(0.131, 0.762)	(0.167, 0.259)
	$\mathscr{A}^{\xi}{}_3$	(0.483, 0.247)	(0.623, 0.566)	(0.290, 0.140)	(0.827, 0.657)	(0.573, 0.837)
	$\mathscr{A}^{\xi}{}_4$	(0.237, 0.964)	(0.926, 0.460)	(0.779, 0.868)	(0.555, 0.735)	(0.783, 0.925)
K_3	$\mathscr{A}^{\xi}{}_1$	(0.288, 0.822)	0.225, 0.677)	(0.373, 0.183)	(0.877, 0.221)	(0.384, 0.213)
	$\mathscr{A}^{\xi}{}_2$	(0.276, 0.174)	(0.753, 0.422)	(0.427, 0.643)	(0.644, 0.336)	(0.517, 0.346)
	$\mathscr{A}^{\xi}{}_3$	(0.724, 0.456)	(0.469, 0.425)	(0.619, 0.421)	(0.528, 0.422)	(0.537, 0.353)
	$\mathscr{A}^{\xi}{}_4$	(0.426, 0.231)	(0.671, 0.421)	(0.473, 0.228)	(0.427, 0.323)	(0.674, 0.266)

Table 4: Combined evaluation matrix

	$\mathscr{C}^{\varsigma}{}_1$	$\mathscr{C}^{\varsigma}{}_2$	$\mathscr{C}^{\varsigma}{}_3$	$\mathscr{C}^{\varsigma}{}_4$	$\mathscr{C}^{\varsigma}{}_5$
$\mathscr{A}^{\xi}{}_1$	(0.8361, 0.2772)	(0.6624, 0.3240)	(0.7844, 0.1455)	(0.8058, 0.2845)	(0.8219, 0.2304)
$\mathscr{A}^{\xi}{}_2$	(0.7469, 0.1813)	(0.7671, 0.4035)	(0.4271, 0.6757)	(0.5760, 0.3505)	(0.4788, 0.3160)
$\mathscr{A}^{\xi}{}_3$	(0.7043, 0.3812)	(0.5169, 0.3925)	(0.6249, 0.2907)	(0.6196, 0.4947)	(0.5472, 0.4289)
$\mathscr{A}^{\xi}{}_4$	(0.4233, 0.3371)	(0.7909, 0.4293)	(0.5569, 0.2467)	(0.4654, 0.4035)	(0.6061, 0.3520)

$$T_{ij}^{(2)} = \begin{pmatrix} 0.4884 & 0.3035 & 0.3849 & 0.4727 & 0.4429 \\ 0.4112 & 0.2645 & 0.0590 & 0.2159 & 0.1628 \\ 0.2948 & 0.1273 & 0.1999 & 0.1654 & 0.1716 \\ 0.1568 & 0.2403 & 0.1645 & 0.1408 & 0.1955 \end{pmatrix}$$

$$T_{ij}^{(3)} = \begin{pmatrix} 0.1008 & 0.0003 & 0.1411 & 0.1776 & 0.1451 \\ 0.0479 & 0.1272 & 0.0004 & 0.0021 & 0.0094 \\ 0.0261 & 0.0106 & 0.0129 & 0.0351 & 0.0006 \\ 0.0001 & 0.1651 & 0.0045 & 0.0027 & 0.0010 \end{pmatrix}$$

Step 4:

Aggregate the FrFvalues $\widetilde{\chi}_{ij}$ for each alternative $\mathscr{A}^{\xi}{}_i$ by the FrFPA$_d$ operator using Equation 0.26 given in Table 5.

$$T_{ij} = \begin{pmatrix} 1 & 0.6109 & 0.3838 & 0.1554 & 0.0874 \\ 1 & 0.4975 & 0.3447 & 0.0196 & 0.0064 \\ 1 & 0.4186 & 0.2255 & 0.0511 & 0.0159 \\ 1 & 0.2691 & 0.1905 & 0.0369 & 0.0099 \end{pmatrix}$$

Step 5:

Table 5: FrF aggregated values $\widetilde{\chi}_i$

$\widetilde{\chi}_1$	(0.791187, 0.257588)
$\widetilde{\chi}_2$	(0.719510, 0.288512)
$\widetilde{\chi}_3$	(0.657383, 0.373734)
$\widetilde{\chi}_4$	(0.568093, 0.366204)

Compute the score for all aggregated values $\widetilde{\chi}_i$.

$$\mathscr{O}^{\varsigma}(\widetilde{\chi}_1) = 0.739087$$

$$\mathscr{O}^{\varsigma}(\widetilde{\chi}_2) = 0.674236$$

$$\mathscr{O}^{\varsigma}(\widetilde{\chi}_3) = 0.615944$$

$$\mathscr{O}^{\varsigma}(\widetilde{\chi}_4) = 0.567115$$

Step 6:
Ranks according to score values.

$$\widetilde{\chi}_1 \succ \widetilde{\chi}_2 \succ \widetilde{\chi}_3 \succ \widetilde{\chi}_4$$

So,

$$\mathscr{A}^{\xi}{}_1 \succ \mathscr{A}^{\xi}{}_2 \succ \mathscr{A}^{\xi}{}_3 \succ \mathscr{A}^{\xi}{}_4$$

$\mathscr{A}^{\xi}{}_1$ is best alternative among all other alternatives.

Conclusion

Using MSDs and NMSDs to handle ambiguity in the data, the current endeavor employs FrFNs. The FrF framework expands upon the IFS paradigm. Considering stringent priority orders, we introduced the concepts of FrFPA$_d$ operator and FrFPG$_d$ operator with PDs. Numerous PD hypotheses have been exhaustively investigated, and they will be beneficial when combining numerous FrF data. A group MCDM approach based on the proposed prioritized AOs has been developed under the FrF framework. The proposed technique is illustrated with an analogy, and the results of the methodology are contrasted to those of several existing AOs. The effect of PDs on aggregated outcomes is otherwise comprehensively explained. Moreover, the influence of PDs on outcomes makes the proposed solution more robust because the DM can select the PD vector based on his or her priorities and the problem's complexity. We implement the proposed group MCDM methodology to a case study regarding the selection of agricultural land. Some functional applications of the proposed work in imprecise inferences could be explored in the future. In addition to decision-making, medical diagnosis, pattern recognition, computational intelligence, and artificial intelligence, we would employ the suggested AOs and MCDM methodology. In the future, we will also concentrate on developing methods for objectively acquiring PD.

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